11. Linear Programming

- Problems which seek to maximise (or minimise) a function subject to certain constraints are called optimisation problems.
- A Linear Programming Problem (L.P.P.) is the one that is concerned with finding the optimal value (maximum or minimum value) of a linear function of several variables (called objective function), subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called constraints). The variables are sometimes called the decision variables.

  For example: The following is an L.P.P.

  Maximize $Z = 10x + 12y$

  Subject to the following constraints:

  $5x + 3y \leq 30$ ... (1)

  $x + 2y \geq 2$ ... (2)

  $x \geq 0, y \geq 0$ ... (3)

  In this L.P.P, the objective function is $Z = 10x + 12y$

  The inequalities (1), (2), and (3) are called constraints.

- The common region determined by all the constraints including the non-negative constraints $x \geq 0, y \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem. The region outside this feasible region is called infeasible region.
- Points within and on the boundary of the feasible region represent feasible solutions of the constraints. Any point outside the feasible region is an infeasible solution.
- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

- **Fundamental Theorems for Solving Linear Programming Problems:**

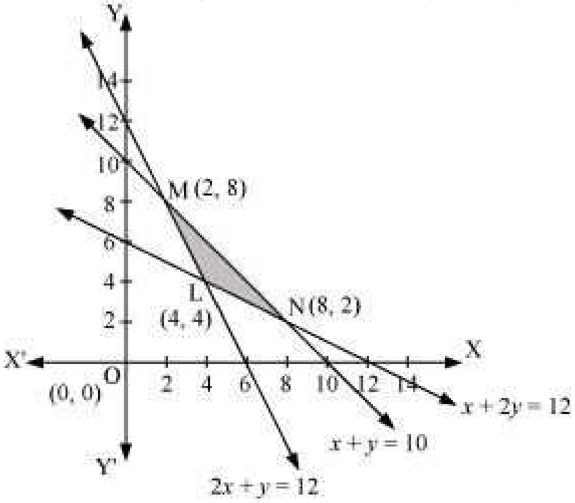
Theorem 1: Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value, where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point of R.

Example: Minimise and Maximise $Z = 35x + 45y$
Subject to constraints
$x + 2y \geq 12$ ... (1)
$2x + y \geq 12$ ... (2)
$x + y \leq 10$ ... (3)

$x \geq 0, y \geq 0$ ... (4)

The feasible region determined by the system of constraints is as follows:



The shaded region is the feasible region.
The corner points of the feasible region are L (4, 4), M (2, 8), and N (8, 2).
∴ The optimal value of Z i.e., the maximum or minimum value of Z must occur at either L (4, 4), M (2, 8), or N (8, 2).

- If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, then it must occur at a corner point of R.
- **Corner point method:** This method is used for solving a linear programming problem and it comprises of the following steps:

  Step 1) Find the feasible region of the L.P.P. and determine its corner points.
  Step 2) Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.
  Step 3) If the feasible region is bounded, then M and m respectively are the maximum and minimum values of the objective function.
  If the feasible region is unbounded:

  - If the open half plane determined by $ax + by > M$ has no point in common with the feasible region, then M is the maximum value of the objective function. Otherwise, the objective function has no maximum value.
  - If the open half plane determined by $ax + by < m$ has no point in common with the feasible region, then m is the minimum value of the objective function. Otherwise, the objective function has no minimum value.

- If two corner points of the feasible region are both optimal solutions of the same type, i.e. produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.
- A few important linear programming problems are: diet problems, manufacturing problems, transportation problems, and allocation problems.

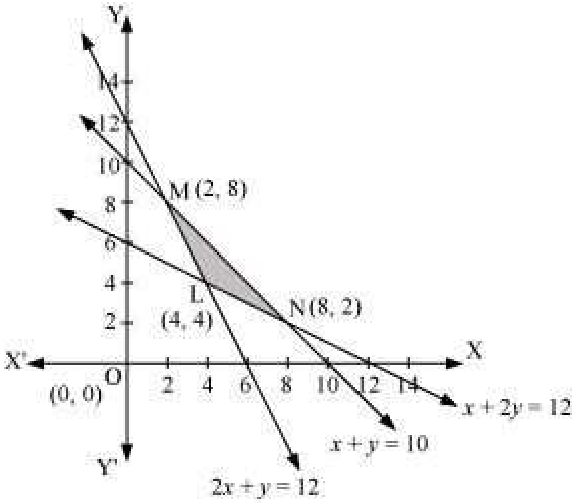
Example 1:
A firm is engaged in breeding goats. The goats are fed on various products grown in the farm. They require certain nutrients, named A, B, and C. The goats are fed on two products P and Q. One unit of product P contains 12 units of A, 18 units of B, and 25 units of C, while one unit of product Q contains 24 units of A,

9 units of B, 25 units of C. The minimum requirement of A and B are 144 units and 108 units respectively whereas the maximum requirement of C is 250 units. Product P costs Rs 35 per unit whereas product Q costs Rs 45 per unit. Formulate this as a linear programming problem. How many units of each product may be taken to minimise the cost? Also find the minimum cost.

Solution:
Let x and y be the number of units taken from products P and Q respectively to minimise the cost.
Mathematical formulation of the given L.P.P. is as follows:
Minimise $Z = 35x + 45y$

Subject to constraints
$12x + 24y \geq 144$ (constraints on A) $\Rightarrow x + 2y \geq 12$ ... (1)
$18x + 9y \geq 108$ (constraints on B) $\Rightarrow 2x + y \geq 12$ ... (2)
$25x + 25y \leq 250$ (constraints on C) $\Rightarrow x + y \leq 10$ ... (3)
$x \geq 0, y \geq 0$ ... (4)

The feasible region determined by the system of constraints is as follows:



The shaded region is the feasible region.
The corner points are L (4, 4), M (2, 8), and N (8, 2). The value of Z at these corner points are as follows:

Corner point	$Z = 35x + 45y$	
L (4, 4)	320	→ Minimum
M (2, 8)	430	
N (8, 2)	370	

It can be observed that the value of Z is minimum at the corner point L (4, 4) and the minimum value is 320.
Therefore, 4 units of each of the products P and Q are taken to minimise the cost and the minimum cost is Rs 320.
The iso-profit or iso-cost method is a graphical method of finding the optimal solution of a linear programming problem (LPP). In this method, we give a suitable constant value to the objective function and draw the corresponding line of the objective function. If the objective function is of maximisation type, then the initial objective line is called the iso-profit line; whereas if the objective function is of minimisation type, then the initial objective line is called the iso-cost line.

Follow the given steps to solve a LPP by using the iso-profit or iso-cost method:

1. Graph the constraints and determine the feasible region.

2. Draw the initial objective line $Z_1 = ax + by$ that passes through a point lying in the feasible region.

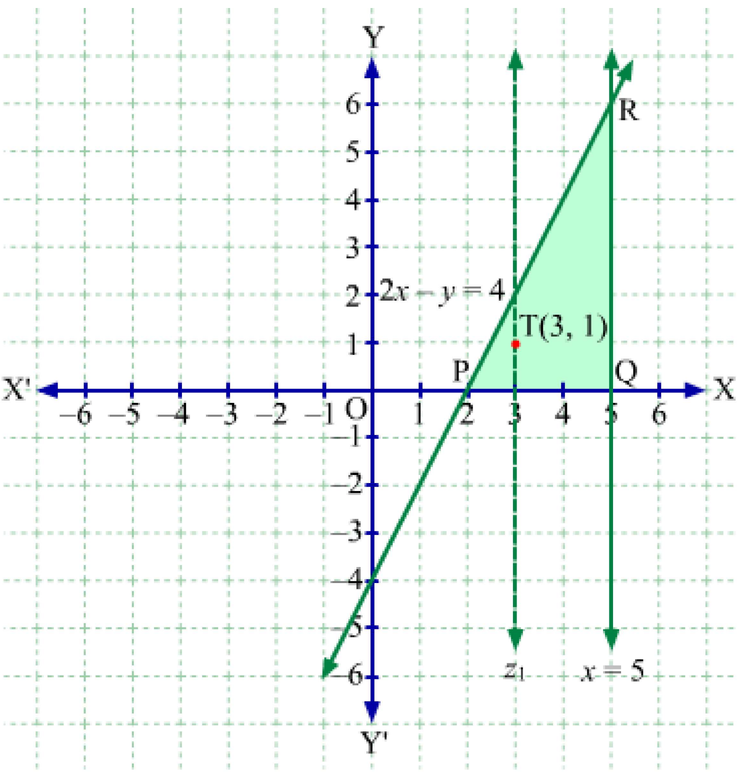
3. If the objective function is of maximisation type, then move the iso-profit line $Z_1 = ax + by$ away from the origin and parallel to itself till the feasible extreme point(s) is/are located. This iso-profit line will give the maximum value of Z.

4. If the objective function is of minimisation type, then move the iso-cost line $Z_1 = ax + by$ towards the origin and parallel to itself till the feasible extreme point(s) is/are located. This iso-cost line will give the minimum value of Z.**1. Infinite Number of Infinite Optimal Solutions**

Some LPPs may have an infinite number of solutions.
To understand these types of LPPs, let us take an example:

Example 1:
Maximise $z = x$ subject to
$x \leq 5; x \geq 0$
$2x - y \geq 4; y \geq 0$

Solution:
The shaded region is the feasible region.



Take a point T3, 1 of the feasible region.
∵ z1=3
The initial iso-profit line is given by:

$x = 3$

Let us move this line away from the origin till the extreme points of the feasible region are located. Thus, the iso-profit line coincides with the line segment QR. Therefore, all points on the line segment QR give the same value of the objective function.

Now, take any point on QR, say, Q(5, 0). It will give the maximum value of the objective function.

∴ Max(z)=5

Hence, the optimal value is unique. It is equal to 5, but there are an infinite number of optimal solutions lying on the line segment QR.

2. Unbounded Solutions

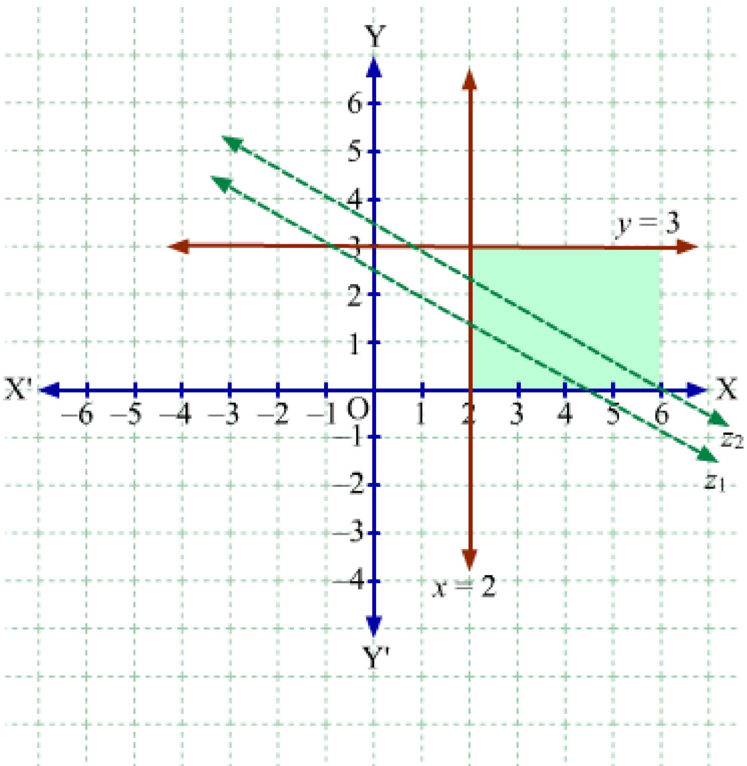
Some LPPs may have unbounded solutions.

To understand these types of LPPs, let us take an example:

Example 2:
Maximise $z = x + 2y$ subject to
$x \geq 2$
$y \leq 3; y \geq 0$

Solution:

The shaded region is the feasible region, as shown in the following figure:



As we move away from the origin, we never get the extreme points of the feasible region. Thus, the objective function takes infinite values for the decision variables lying in the feasible region. So, there is no finite maximum value of z. Hence, the LPP is said to have an unbounded solution.

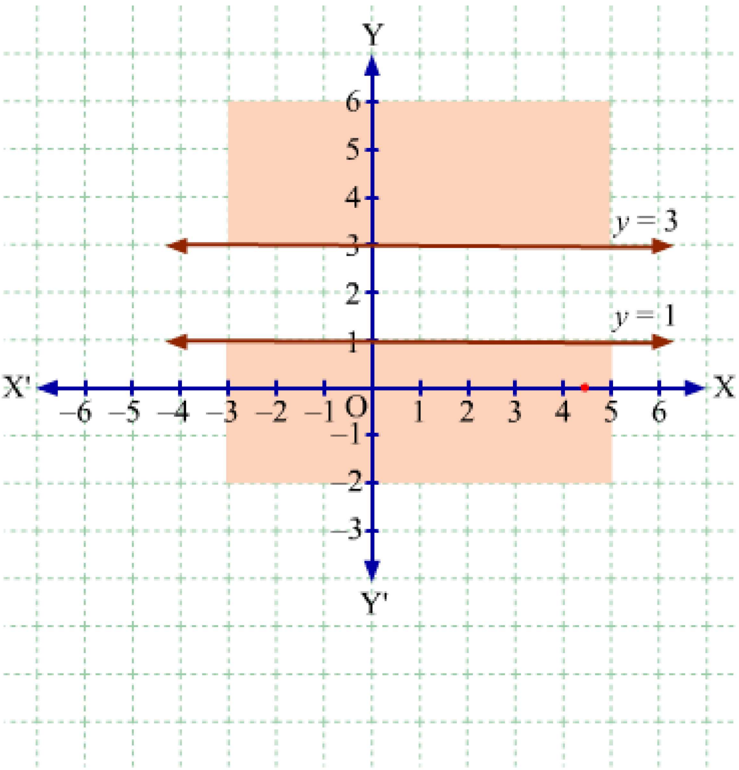
3. Infeasible Solutions

There are LPPs that do not have any solutions. To understand this case, let us take an example:

Example 3:
Maximise $z = 8x - y$ subject to

$y \le 1$
$y \ge 3$
$x, y \ge 0$

Solution:



We observe that there is no feasible region satisfying all constraints. Hence, the given LPP has no solution.